Choices and Intervals

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joint work with

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Related models

Random structures formed by adding objects **one after the other** according to some random rule. Examples:

1. balls-and-bins model: n bins, place balls one after the other into bins, for each ball choose bin uniformly at random (maybe with size-biasing)
2. random graph growth: n vertices, add (uniformly chosen) edges one after the other.
3. interval fragmentation: unit interval $[0, 1]$, add uniformly chosen points one after the other $\rightarrow$ fragmentation of the unit interval.

Extensive literature on these models.

Power of choices

Aim: Changing behaviour of model by applying a different rule when adding objects

1. balls-and-bins model: n bins, at each step choose two bins uniformly at random and place ball into bin with fewer/more balls.
   Azar, Broder, Karlin, Upfal '99; D'Souza, Krapivsky, Moore '07; Malyshkin, Paquette '13
2. random graph growth: n vertices, at each step uniformly sample two possible edges to add, choose the one that (say) minimizes the product of the sizes of the components of its endvertices.
   Achlioptas, D'Souza, Spencer '09; Riordan, Warnke '11+'12
3. interval fragmentation: unit interval $[0, 1]$, at each step, uniformly sample two possible points to add, choose the one that falls into the larger/smaller fragment determined by the previous points.
   $\rightarrow$ **this talk**

Balls-and-bins model

n bins, place n balls one after the other into bins.

- Model A: For each ball, choose bin uniformly at random.
- Model B: For each ball, choose two bins uniformly at random and place ball into bin with more balls.
- Model C: For each ball, choose two bins uniformly at random and place ball into bin with fewer balls.

How many balls in bin with largest number of balls?

- Model A:
- Model B:
- Model C:

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- Model C: $O(\log\log n)$

Ψ-process: definition

X: random variable on $[0,1]$, $\Psi(x) = P(X \leq x)$.

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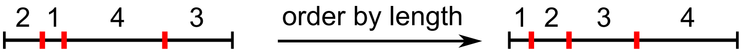

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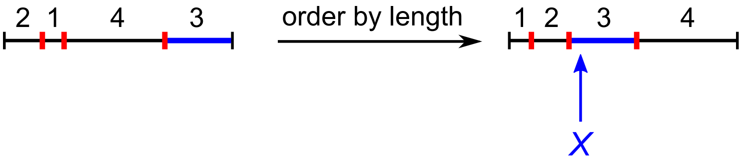

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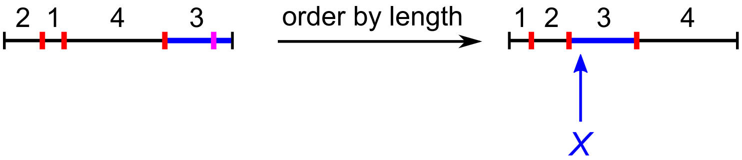

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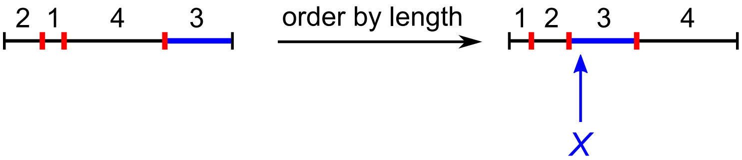
Ψ-process: examples



X: random variable on $[0, 1]$, $\Psi(x) = P(X \leq x)$.

- $\Psi(x) = x$: uniform process
- $\Psi(x) = \mathbf{1}_{x \geq 1}$: *Kakutani process*
- $\Psi(x) = x^k$, $k \in \mathbb{N}$: max-k-process (maximum of k intervals)
- $\Psi(x) = 1 - (1 - x)^k$, $k \in \mathbb{N}$: min-k-process (minimum of k intervals)

Main result

$I_1^{(n)}, \dots, I_n^{(n)}$: lengths of intervals after step n.

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{n \cdot I_k^{(n)}}$$

Main theorem

Assume Ψ is continuous + polynomial decay of $1 - \Psi(x)$ near $x = 1$.

1. μ_n (weakly) converges almost surely as $n \to \infty$ to a deterministic probability measure μ^Ψ on $(0, \infty)$.
2. Set $F^\Psi(x) = \int_0^x y \, \mu^\Psi(dy)$. Then F^Ψ is C^1 and

$$(F^\Psi)'(x) = x \int_x^\infty \frac{1}{z} \, d\Psi(F^\Psi(z)).$$

Properties of limiting distribution

Write $\mu^\Psi(dx) = f^\Psi(x)\,dx$.

max-k-process ($\Psi(x) = x^k$)

$$f^\Psi(x) \sim C_k \exp(-kx), \qquad \text{as } x \to \infty.$$

min-k-process ($\Psi(x) = 1 - (1 - x)^k$)

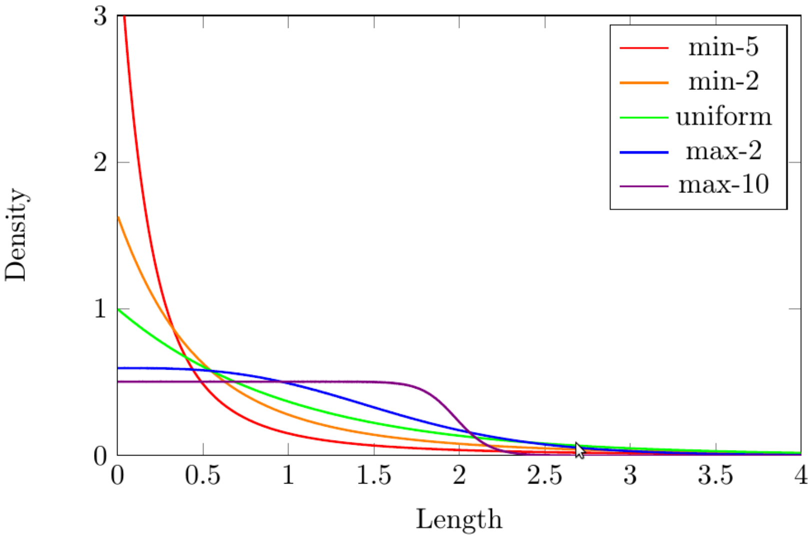
$$f^\Psi(x) \sim \frac{c_k}{x^{2+\frac{1}{k-1}}}, \qquad \text{as } x \to \infty.$$

convergence to Kakutani (cf. Pyke '80)

If $(\Psi_n)_{n\geq 0}$ s.t. $\Psi_n(x) \to \mathbf{1}_{x\geq 1}$ pointwise, then

$$f^{\Psi_n}(x) \to \frac{1}{2}\mathbf{1}_{x\in[0,2]}, \qquad \text{as } n \to \infty.$$

Properties of limiting distribution (2)



Proof of main theorem: the stochastic evolution

Embedding in continuous time: points arrive according to Poisson process with rate e^t.
N_t: number of intervals at time t
$l_1^{(t)}, \ldots, l_{N_t}^{(t)}$: lengths of intervals at time t.
Observable: size-biased distribution function

$$A_t(x) = \sum_{k=1}^{N_t} l_k^{(t)} \mathbf{1}_{l_k^{(t)} \le x e^{-t}}$$

Then $\boldsymbol{A} = (A_t)_{t\geq 0}$ satisfies the following stochastic evolution equation:

$$A_t(x) = A_0(e^{-t}x) + \int_0^t (e^{s-t}x)^2 \left[\int_{e^{s-t}x}^{\infty} \frac{1}{z} d\Psi(A_s(z)) \right] ds + M_t(x),$$

for some centered noise M_t.
Claim: A_t converges almost surely to a deterministic limit as $t \to \infty$.

Deterministic evolution

Let $\boldsymbol{F} = (F_t)_{t\geq 0}$ be solution of

$$F_t(x) = F_0(e^{-t}x) + \int_0^t (e^{s-t}x)^2 \left[\int_{e^{s-t}x}^{\infty} \frac{1}{z} d\Psi(F_s(z))\right] ds$$
$$=: \mathscr{S}^{\Psi}(\boldsymbol{F})_t.$$

Define the following norm:

$$\|f\|_{x^{-2}} = \int_0^{\infty} x^{-2} |f(x)|\, dx.$$

Lemma

Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be solutions of the above equation. For every $t \geq 0$,

$$\|F_t - G_t\|_{x^{-2}} \leq e^{-t} \|F_0 - G_0\|_{x^{-2}}.$$

In particular: $\exists! F^{\Psi} : F_t \to F^{\Psi}$ as $t \to \infty$.

Stochastic evolution - stochastic approximation

Problem

Cannot control noise M_t using the norm $\|\cdot\|_{x^{-2}}$!
$\Longrightarrow$ no quantitative estimates to prove convergence.

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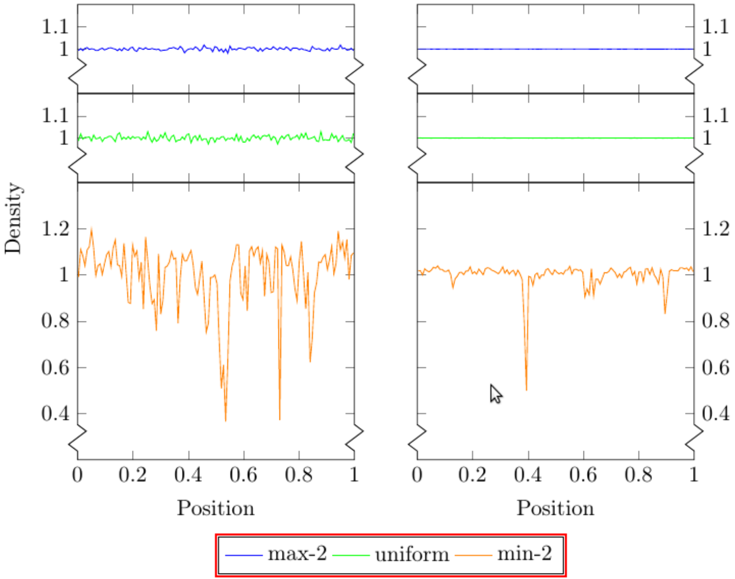
Still possible to prove convergence by Kushner–Clark method for stochastic approximation algorithms.

1. Shifted evolutions $\boldsymbol{A}^{(n)} = (A^{(n)}_{t-n})_{t\in\mathbb{R}}$. Show: almost surely, the family $(\boldsymbol{A}^{(n)})_{n\in\mathbb{N}}$ is precompact in a suitable functional space.
2. Show $\mathscr{S}^{\Psi}$ is continuous in this functional space.
3. Show $\boldsymbol{A}^{(n)} - \mathscr{S}^{\Psi}(\boldsymbol{A}^{(n)}) \to 0$ almost surely as $n \to \infty$.

This entails that every subsequential limit $\boldsymbol{A}^{(\infty)}$ of $(\boldsymbol{A}^{(n)})_{n\in\mathbb{N}}$ is a fixed point of $\mathscr{S}^{\Psi}$. By previous lemma: $\boldsymbol{A}^{(\infty)} \equiv F^{\Psi}$.

Note: precompactness shown by entropy bounds, already used by Lootgieter '77; Slud '78.

Open problem: empirical distribution of points



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